Natural language and "mathematics languages": Intuitive models and stereotypes in the mathematics classroom

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Abstract. *Through episodes taken from various research studies in mathematics education carried out over the years by one of the authors, we bring evidence of the interference between natural language and specific language. Within a semiotic perspective we show how and why students' learning experience entails the emergence of intuitive models and stereotypes in mathematics classroom. The notion of didactical contract allows us to interpret students' stereotypical behaviors within the Chevallard's triangle, Knowledge – teacher – student.*

Keywords: natural language and specific language, stereotypes, intuitive models, objectification, semiotic means of objectification, semiotic system, semiotic functions, meaning.

Sunto. *Attraverso episodi tratti da varie ricerche in didattica della matematica, condotte nel corso degli anni da uno degli autori, mettiamo in evidenza l'interferenza tra linguaggio naturale e linguaggio specifico. All'interno di una prospettiva semiotica mostriamo come e perché l'esperienza di apprendimento degli studenti comporti l'emergere di modelli intuitivi e stereotipi nelle lezioni di matematica. La nozione di contratto didattico ci consente di interpretare i comportamenti stereotipati degli studenti all'interno del triangolo di Chevallard, Sapere - insegnante - studente.*

Parole chiave: linguaggio naturale e linguaggio specifico, stereotipi, modelli intuitivi, oggettivazione, mezzi semiotici di oggettivazione, sistema semiotico, funzioni semiotiche, significato.

Resumen. *A través de episodios tomados de diversas investigaciones en educación matemática, realizadas a lo largo de los años por uno de los autores, mostramos algunas pruebas de la interferencia entre el lenguaje natural y el lenguaje específico. Dentro de una perspectiva semiótica, mostramos cómo y por qué la experiencia de aprendizaje de los estudiantes implica la aparición de modelos intuitivos y estereotipos en los cursos de matemática. La noción de contrato educativo nos permite interpretar los comportamientos estereotípicos de los estudiantes dentro del triángulo de Chevallard, Saber - profesor - alumno.*

Palabras clave: lenguaje natural y lenguaje específico, estereotipos, modelos intuitivos, objetivación, medios semióticos de objetivación, sistema semiótico,

funciones semióticas, significado.

1. Introduction

We would like to start the article with the following episode: *I leave this task to the real teachers*.

Seventh grade, students aged 12 or 13. The idea is to bring to light stereotypes (primarily linguistic) and conditionings in mathematical practice at school. I fall back on the "trick" of "Pretend that you are …" (D'Amore & Sandri, 1996). The task proposed is: "Pretend that you are a primary school teacher. … You want to explain to your third grade (8-year-old) students that the area of a rectangle is found by doing base times height". Very many students limit themselves to writing a formula, others explain that the base is *b* and the height is *h*, while others confuse rectangle with triangle. Very few agree to write things out, really getting into the part. One girl did so, however, producing this little masterpiece:

> I don't think I'm up to pretending that I'm a primary school teacher, but I can always try: there's always a first time. Above all, if I really were to be a teacher, I would be very spontaneous and pleasant, so that dialogue with my students could be simple and direct. I would like to have a friendly and amusing relationship, so if I actually had to explain how to find the area of a rectangle, in view of my love of sweet things, I would think of the rectangle as a slab of chocolate. I have tried, but without success. I'm not able to explain that the area of a triangle is found by doing base times height. I leave this task to the real teachers.

This episode is intriguing because apparently little mathematics is involved but nevertheless it tells us a lot about the learning of mathematics. The metacognitive awareness of this pupil is outstanding. Although reluctant, she gives a try in pretending she is a real teacher revealing self-confidence. She highlights her emotional needs for an effective learning of mathematics: spontaneity, pleasantness, communication, simplicity and directness, personal likes. She ends up with a "metacognition in the negative" since she perceives that she is unable to explain how to find the area of a triangle (rectangle) and doesn't feel unease in expressing her difficulty. Above all, this short extract shows the pupil's demand of emotional well-being and personal involvement in his/her learning trajectory.

Luis Radford describes learning as a process of *objectification* that consists in "actively and imaginatively endowing the conceptual objects that the student finds in his/her culture with meaning" (Radford, 2008, p. 223). As pointed out by Godino and Batanero (1994) and Radford (2006), meaning is a double-sided construct consisting of a *personal/individual* meaning and an *institutional/cultural* meaning that are somehow distinguishable but inseparable, like two sides of the same coin. In his/her learning path the student embodies the interpersonal and general meaning of mathematical concepts.

Endowing conceptual objects with meaning clashes against an intrinsic difficulty that lies in the special ontological status of mathematics, whose objects do not allow ostensive references. The only way to access mathematical concepts is to have recourse to a wide range of semiotic representations that theoretical perspectives in mathematics education have positioned with different understandings within their system of principles: Duval’s functional and structural approach (Duval, 1995), Radford’s theory of knowledge objectification (Radford, 2008), Godino’s onto semiotic approach (D’Amore & Godino, 2006; D’Amore & Fandiño Pinilla, 2017), and Arzarello’s semiotic bundle (Arzarello, 2006), just to quote a few examples. Basically, in all perspectives the signs can play a *representational* role or serve as *mediators* of personal and socially shared activities. D’Amore and Fandiño Pinilla (2008) and Santi (2011) have shown that the two perspectives are complementary to each other and can be effectively coordinated to frame mathematical cognition.

In their learning process, students have to handle a complex implementation of signs, that requires to properly *coordinate* several semiotic resources. Furthermore, they have to overcome Duval’s (1995) *cognitive paradox* that leads them to identify mathematical concepts with their representations. D’Amore (2001) accounts for the students’ lack of personal involvement in the learning process, by acknowledging the cognitive and emotional strain entailed in the attempt to access mathematical knowledge. Often their need to endow cultural objects with meaning diverts from the cultural meaning expected by the teacher and the school institution, by clinging to inappropriate *intuitive models* (Fischbein, 2002), *stereotypes* and the *didactical contract* (Brousseau, 1997).

In section 2, we propose a theoretical framework that allows us to analyze the interplay between natural language and mathematical language within a semiotic perspective.

In section 3, we analyze episodes taken from various research studies in mathematics education carried out over the years by one of the authors. Our aim is to highlight the interplay between natural language and *other semiotic systems* when students shift to higher layers of generality by addressing specific mathematical languages.

In section 4, we draw some conclusions of our study.

2. Theoretical framework

As we mentioned in section 1, having recourse to semiotic representations is the only way to access mathematical objects. There are two complementary approaches towards signs. A socio-cultural approach based on Vygotskian *activity theory* stances and a *structural and functiona*l approach.

We present both approaches describing the basic features of Radford's theory of knowledge objectification and Duval's structural and functional approach.

2.1. The theory of knowledge objectification

The theory of knowledge objectification (TKO) pivots around the notion of mediated reflexive activity that conceives *thinking* as "a mediated reflection in accordance with the form or mode of the activity of individuals" (Radford, 2008, p. 218):

- *activity* refers to the individual and social agency towards shared goals, significant problems, operations, labor etc., within a cultural dimension that provides a system of beliefs, conceptions about truth, methods of inquiry, acceptable forms of knowledge;
- *reflection* refers to the dialectical movement of the individual consciousness between his personal thinking, interpretations, emotions and feelings, perceptions and a historically and culturally constituted reality;
- *mediation* refers to the artifacts that, in a Vygotskian sense, are constitutive and consubstantial to thinking since they allow us to carry out activity, i.e. they mediate activity; within TKO the system of artifacts that carry out activity are termed as the *territory of artifactual thought* and it includes objects, artifacts, gestures, natural language, symbolic language, icons, drawings etc.

Advocating a pragmatic ontology, in TKO *mathematical objects* are "fixed patterns of reflexive human activity incrusted in the ever-changing world of social practice mediated by artifacts" (Radford, 2008, p. 222). In this view, mathematical objects lose any intrinsic, a priori, realistic nature. Nevertheless, as fixed patterns of mediated reflexive activity they acquire, within the *cultural-historical dimension,* a form of *ideal existence*:

> "Ideality" is rather like a stamp impressed on the substance of nature by social human life activity, a form of the functioning of the physical thing in the process of this activity. So, all the things involved in the social process acquire a new "form of existence" that is not included in their physical nature and differs from it completely – [this is] their ideal form. (Ilyenkov, 1977, p. 86)

Radford (2016, p. 3) conceptualizes activity in terms of *joint labor*:

> The idea of joint labor seeks to restore to activity its most precious ontological force, namely, the dynamic locus where human existence creates and recreates itself against the backdrop of culture and history. Yet, with its utilitarian and consumerist orientation, contemporary mathematics classroom activity tends to produce and reproduce alienated students. It is argued that the search for non-alienating classroom activity requires a reconceptualization of the classroom's forms of human collaboration and its modes of knowledge production.

(See also: D'Amore, 2015, 2018).

We can say that, after their emergence as fixed patterns of reflexive human activity, mathematical objects become objects of knowledge for students involved in the learning path. TKO defines *learning* as a particular form of mediated reflection, a process of *objectification*:

> An opening movement towards others and the objects of culture. (…) To learn is not merely to acquire something in the corrupted sense of possessing it or mastering it, but to go to culture to find something in it. This is why the outcome of the act of learning is not the construction, re-construction, re-production, re-invention or mastering of concepts: its true outcome is to be found in the fact that, in this encounter with the other and cultural objects, the seeking individual *finds herself*. This creative process of finding or noticing something (a dynamic target) is what I have termed elsewhere a process of *objectification* (Radford, 2002). (Radford, 2008, p. 222)

The artifacts that mediate this special form of reflexive activity are called *semiotic means of objectification.* They are bearers of the historical and cultural development of mathematics and they allow students to transform interpersonal and ideal concepts into embodied objects belonging to their space-time and emotional experience.

Semiotic means of objectification determine the mode of existence of mathematical objects in the pupils' experience, i.e. they determine how the intentional "arrow" attends such objects. Referring to Husserl (1913/1931) they intertwine the *noetic-noematic phenomenological layers* that altogether result in the full meaning of the mathematical object. For example, we can deal with the circle through the kinesthetic movement of the compass, the definition in natural language, and using a second-degree equation in the algebraic symbolism. The TKO allows us to outline *levels of generality* (Radford, 2004) at which the student objectifies the mathematical concept. The level of generality specifies the degree of embodied experience involved in the reflection mediated by a particular semiotic means of objectification. Recalling the example mentioned above, the compass mediates the circle with a lower level of generality with respect to the second-degree equation. The demand of higher levels of generality, as the individual and cultural meanings converge, obliges the pupil to live a rupture with his/her embodied experience that can bewilder him and lower his personal implication and involvement in the learning process (Radford, 2003).

The role of natural language as a semiotic means of objectification is the turning point in bridging the gap between the embodied experience of the pupil and the interpersonal meaning of the cultural object. Some research (Radford, 2000, 2002, 2004) in this topic has shown how in the generalization of patterns the deictic and generative use of natural language triggers and enhances the shift from the sensorimotor experience to the algebraic symbolism. Exposing the students directly to the algebraic language would result in a shallow learning. Furthermore, as we reach higher levels of generality natural language allows us to keep the relation with the core of meaning that lies in the individual embodied experience. Natural language plays a key role in driving movement, organizing activity in time and space, in triggering rhythm, in singling out and individualizing objects, in activating and supporting schemes. This broad set of possibilities is the inerasable basis for the recognition of operational invariants, thus accessing higher layers of generality. A thorny issue is the relation between natural language and the use of the specific language of mathematics. It requires attention and awareness on the part of the teacher. The use of the specific language of mathematics is a learning objective that allows the student to reach a further layer of generality. It is achieved when embodied meaning has a solid basis to sustain the leap to specific mathematical language that objectifies definitions, generalizations, algorithms, inferential thinking etc. Without an underlying significant reflexive activity in the student's personal experience, the use of specific language can hinder the learning of mathematics. Furthermore, the specific language of mathematics has a semantic density (D'Amore, 1999) that can disembody meaning just as it happens with symbolic language. In this situation the combined use of symbolic language and specific language can foster an unbridgeable gap between the individual meaning of the student and the cultural one, thus entailing a lack of personal implication and the emergence of intuitive models (Fischbein, 1992, 2002) and the didactical contract (Brousseau, 1997). For an in-depth discussion of this topic we refer the reader to D'Amore (1999, pp. 251–261).

We analyzed the TKO within a pragmatist ontology of mathematics. D'Amore (2003) provides a detailed account of realist and pragmatist theories and concludes that there is not a definite boundary between the two perspectives. Ullmann (1962) highlights two complementary features that characterize the development of mathematical objects: the *operational phase* and the *referential phase*. On the one hand mathematical objects and their meaning emerge from and are objectified by reflexive activity, on the other hand it is necessary to linguistically refer to the entities that emerge from such activity. The dual nature of mathematical objects – as patterns of activity and as existing ideal entities in the culture – implies that also meaning and semiotics have a dual nature. We therefore need a semiotic perspective that accounts for the need, in the referential phase, to nominalize and transform signs in order to create relations, generalize, carry out calculations and proofs. Raymond Duval (1995, 2006, 2008) introduced semiotics in mathematical thinking and learning and devised a structural and functional approach to the use of signs.

2.2. Duval's structural and functional approach

In the referential phase that we are addressing in this paragraph, Duval's

(1995) approach could be framed within a realistic theory of meaning (D'Amore, 2003). In fact, according to Duval, every mathematical concept refers to *objects* that do not belong to our concrete experience. In mathematics, *ostensive referrals are impossible*, therefore every mathematical concept intrinsically requires working with semiotic representations, since we cannot display "objects" that are directly accessible.

The lack of ostensive referrals led Duval to assign a constitutive role in mathematical thinking to the use of representations belonging to specific semiotic registers. From this point of view, he claims that there isn't *noetics without semiotics*. "The special epistemological situation of mathematics compared to other fields of knowledge leads to bestow upon semiotic representations a fundamental role. First of all, they are the only way to access mathematical objects" (Duval, 2006).

The peculiar nature of mathematical objects allows us to outline a specific cognitive functioning that characterizes the evolution and the learning of mathematics. The cognitive processes that underlay mathematical practice are strictly bound to a complex semiotic activity that involves the coordination of at least two representation registers. We can say that conceptualization itself, in mathematics, can be identified with this complex coordination of several representation registers.

A semiotic system is devised by (Duval, 2006; Ernest, 2006):

- a *set of basic signs* that have a meaning only when *opposed to or in relation with* other *basic* signs (for example the decimal numeration system);
- a set of o*rganizing rules* for the production of signs from the *basic* ones and for the transformation of signs;
- an *underlying meaning* deriving resulting from the relation of the *basic* signs that form structured semiotic representations.

A *representation register* is a semiotic system that also accomplishes the functions of *communication*, *objectification* and *treatment* (Duval, 1996).

D'Amore (2001) identifies conceptualization with the following *semiotic functions*, which are specific for mathematics:

- c*hoice of the distinctive traits* of a mathematical object;
- *treatment,* i.e. the transformation of a representation into another representation of the same semiotic register;
- *conversion,* i.e. the transformation of a representation into another representation of another semiotic register.

The very combination of these three "actions" on a mathematical object can be considered as the "construction of knowledge in mathematics". But it is not spontaneous nor easily managed and represents the cause for many difficulties in the learning of mathematics when students struggle with the *cognitive paradox.* (See also: D'Amore, Fandiño Pinilla, Iori, & Matteuzzi, 2015;

D'Amore, Fandiño Pinilla, & Iori, 2013).

Mathematical objects are recognized as invariant entities that bind different semiotic representations, when treatment and conversion transformations are performed, and as such they cannot be referred to directly. Duval identifies the specific cognitive functioning of mathematics with the coordination of a variety of representation registers. Both the development of mathematics as a field of knowledge and its learning are accomplished through such a specific cognitive functioning.

Duval goes beyond Frege's classical *semiotic triangle* (*sinn-bedeutung-zeichen*) and identifies meaning with the couple (*sign-object*), i.e. a relationship between a sign and an object it represents. The sign becomes a rich structure that condenses both the semiotic representation (*zeichen*) and the way (*sinn*) the semiotic expression offers the object according to the underlying meaning of the semiotic structure. Meaning therefore has a twofold dimension:

- *sinn*, the way a semiotic representation offers a mathematical object;
- *bedeutung,* the reference to an inaccessible mathematical object (D'Amore, 2010).

Meaning making processes and learning require to handle different *sinn* that are networked through semiotic transformations without losing the common *bedeutung* to the invariant mathematical object.

While in the operational phase the language plays a prominent role in sustaining the leap to higher levels of generality, in the referential phase the language sustains the coordination of representation registers via treatment and conversion.

In fact, natural language is a special and more complex semiotic system. Basically, it has 4 *discursive functions* (Duval, 1995, p. 91) that characterize it as a language:

- the *referential function* that allows us to designate an object;
- the *apophantic function* that allows us to say something on the objects we designate under the form of complete statements;
- the *discursive expansion function* that allows us to connect these statements in a coherent way;
- the *discursive reflexivity function* that underlines the validity, the mode and the status given to the expression by those who produce it.

The discursive functions of natural language are responsible for an appropriate control of the semiotic functions at a cognitive and metacognitive level. The key-players in the learning environment have to enact an aware understanding and control of the relation between:

- the spontaneous and narrative use of natural language;
- the specialized use of natural language;

- other semiotic registers used for definitions, deductive reasoning, algorithms etc.

On the one hand in TKO, when synchronically used with other semiotic means of objectification natural language allows students to move along the different layers of generality in which the mathematical object is stratified. On the other hand within a structural and functional approach, natural language, in diachronic transformations of signs, allows students to access ideal entities and to deal with mathematics as an observational science:

> It has long been a puzzle how it could be that, on the one hand, mathematics is purely deductive in its nature, and draws its conclusions apodictically, while on the other hand, it presents as rich and apparently unending a series of surprising discoveries as any observational science. Various have been the attempts to solve the paradox by breaking down one or other of these assertions, but without success. The truth, however, appears to be that all deductive reasoning, even simple syllogism, involves an element of observation; namely, deduction consists in constructing an icon or diagram the relations of whose parts shall present a complete analogy with those of the parts of the object of reasoning, of experimenting upon this image in the imagination, and of observing the result so as to discover unnoticed and hidden relations among the parts. (…) As for algebra, the very idea of the art is that it presents formulae which can be manipulated, and that by observing the effects of such manipulation we find properties not to be otherwise discerned. In such manipulation, we are guided by previous discoveries which are embodied in general formulae. These are patterns which we have the right to imitate in our procedure, and are *the icons par excellence* of algebra. (Peirce, 1931-1958, 3.363)

The teaching design cannot underestimate the need of personal meaning that drives the student's activity both when learning at school and in his every-day experience. There are two basic constitutive elements that contribute to personal meaning:

- *operational invariants* of schemata;
- a system of *convictions and interpretations.*

When teaching fails in aligning the cultural meaning with the personal one, the student accomplishes his need of meaning by having recourse to operational invariants, enhanced by beliefs and interpretations, that make him feel self-confident and self-effective in a situation of cognitive and emotional dismay. Mathematics education refers to the operational invariants as *intuitive models* because of the sense of globality, immediacy and self-evidence they convey. The system of beliefs and interpretations that intertwine mathematical knowledge, teacher and pupil is referred as *didactical contract*.

An appropriate use of natural language provides students with the cognitive and metacognitive strength to bear the strain in accessing mathematical meaning that requires to deal with a broad and composite semiotic activity. If it is disregarded because considered not rigorous nor mathematical or biased by inappropriate uses of specific language, the student can turn to inappropriate intuitive models, stereotypes and the didactical contract.

3. Students' episodes

We present accounts of episodes taken from various research studies in mathematics education carried out over the years by Bruno D'Amore and his collaborators.

Episode 1. "Like that, I understand"

At the age of 14, Italian children make an important "choice" for the future course of their studies: having finished middle school, and therefore having completed their compulsory schooling, they "decide" in which upper school they continue their education. Giovanni (I will use this fictitious name for him) is one of the many students who have been disheartened by middle school; he certainly does not shine because of his intelligence (whatever that word means) or his intuition (ditto), but he is a person who stays quietly in his seat. In middle school, especially in mathematics, he was enduring various attacks: the fact of not knowing how to solve "easy" equations, or how to calculate "simple" expressions in an orderly fashion; above all, the fact that he got lost when faced with the problem of "translating" from natural to algebraic language caused him to be branded as scholastically incapable. His destiny is sealed: he is not worthy of the luxurious desks of the grammar schools or the technical high schools and he will have to be content with the benches of a professional institute. His humble parents do not even understand exactly what has happened, but Giovanni is satisfied. He despises formalisms; above all he wants to work with his hands. All his friends respect him because at the age of 12 he knows how to take his moped apart into an infinite number of pieces and put it back together again, and because he is the only one who has a girlfriend. Going to a professional institute means finally being able to work with engines, tools and machines; goodbye cursed compositions, goodbye monstrous expressions. But, alas, he realizes very quickly that things are not exactly like that. After the first few days, his mathematics teacher, introduces polynomials to Giovanni and the whole class (in which the Giovannis are many, almost the entire group). They have to compute sizable products, without quite knowing why, take out common factors, apply rules. Giovanni is in trouble, and with him all the Giovannis. Above all he is stymied by the language used by the teacher. This language vaguely resembles Italian, but so to speak, it is more compressed (we would say: terse, condensed in its expression, with an unambiguous syntax) … The teacher turns to us seeking help. We tell her explicitly that the choice of this mathematics topic does not seem a great idea; it is not suitable for those students, but according to the compulsory government curriculum she has to develop it! We state, once and for all, that this teacher and those like her are not to blame. The blame lies with those who do not provide these teachers with cultural alternatives, thereby making them believe that polynomials etc. are the only mathematics that exists. Having made that clear, we suggest the teacher a "trick" that has worked very well elsewhere. That is, consider how mediaeval school books – those written in the vernacular – present the rules of algebra, explain the procedures, present the problem statements. On other occasions, by comparing those languages with the one we use now, the children have recognized that, in fact, our symbolism and, more generally, our method of doing mathematics, is very much simpler. We ourselves go into the class and experience the following episode. We present mediaeval abaci, Greek geometrical algebra and especially algebraic symbolism that is wholly verbal (rhetorical algebra), and we wait for the reaction obtained in earlier testing. Not in this occasion. This time Giovanni, *that* Giovanni, blurts out exactly these words: "But why don't they do it like that anymore? Like that, I understand!".

Giovanni has been exposed to symbolic language without objectifying algebra at lower levels of generality. He missed the chance to deal with mathematical objects within his embodied space-time and kinesthetic experience, therefore he gave up his personal involvement in learning algebra.

Natural language offers Giovanni the opportunity to fill the gap between his personal need for embodied meaning and the cultural interpersonal meaning objectified by symbolic language. The introduction of rhetorical algebra on the part of the researcher provides the student with the constitutive processes of algebraic thinking. Despite the redundancy and heaviness of rhetorical language, thanks to its generative and deictic potentials, Giovanni is able to recognize the operational invariants in his personal meaning made of space-time experience, movement, feelings and emotions. At this point, the use of symbolic language can prompt the leap to higher levels of generality, by addressing treatment and conversion operations. The *referential, apophantic* and *discursive expansion* functions of natural language drive the basic semiotic operations. They allow to refer to the algebraic entities, describe and characterize them, and embed semiotic transformations in a coherent and rational discourse.

Natural language and mathematics: there is still much to consider in this relationship. Algebraic symbolism as an objective shared by society. Epistemological obstacles connected to the specific language of a discipline and to the pseudo-natural language in which the discipline is carried out or presented to the students. How many profound observations Giovanni gives us!

Episode 2. "I divided the friends among the biscuits"

Lugo di Romagna, an important agricultural center in the province of

Ravenna, Italy. A test concerning intuitive models (Fischbein, 1992) of multiplication and division, which are carefully described in D'Amore (1993a, pp. 168–185), contains the following item:

"15 friends buy 5 kg of biscuits; how many kilos of biscuits should each one receive?".

41% of the students at the end of the first year of scientific high school answer with the operation: 15 ÷ 5. An intuitive model arises. It is based on a "wise" arithmetic practice assumed as a parasitic model (Fischbein, 1992), "always divide the larger by the smaller". The experiment is also carried out in primary school, at the end of Grade 5, with an identical answer given by 67% of the students. When the primary school children are interviewed individually, one after the other, no one acknowledges spontaneously that one should (or that one could) perform 5 ÷ 15, unless the 5 kg are transformed into some large number of grams. But those who are interviewed in the first year of scientific high school react differently: all the students claim to have skimmed over this problem statement, whose semantics was sneaky, and they also admit that they were deceived by the fact that the numerical datum 15 came before 5. Someone also goes on to say that it was so easy to perform 15 ÷ 5 and that this has lowered his critical threshold. One of the cleverer students understands straight away, laughs, hits his forehead with his hand and exclaims: "Instead of dividing the biscuits among the friends, I divided the friends among the biscuits".

Fischbein (2002) defines intuitive thinking as an immediate self-evident, global, coercive form of thinking. We always urge, consciously or unconsciously, for intuitive thinking because of its immediacy, self-evidence and globality. It fosters positive emotions and a sense of well-being towards thinking and learning. Its strength can be traced back to embodied activity, which is objectified by sensorimotor experience and movement. Assigning the 15 kg of biscuits among the five friends instead of doing the other way around is perceptually immediate and self-evident. When it is 5 kg of biscuits among the 15 friends the self-evident embodied meaning evaporates. The previous operational invariant for division, with its strong embodied meaning condensed in a parasitic model (Fischbein, 2002), is thus applied. The term "division" assumes a different meaning; it has to reach a higher layer of generality. Having recourse to natural language, it is interesting how the high school student is able to visualize the interaction, in space-time, between the kilos of biscuits and the number of friends. Thereby he is aware that he is addressing the pattern of division where the dividend is greater than the divisor and he is dividing the friends among the biscuits. Such an awareness leads the student to a new objectification of the problem in natural language:

"5 kilos of biscuits must be obtained by multiplying the 15 friends by the kilos of biscuits per student".

The student is ready to use symbolic language: he writes

5 kg = 15 friends × *n* kilos/friend, therefore the kilos per student are 5 divided by 15.

From a structural and functional point of view the congruence between natural language, symbolic arithmetical register and the iconic register in which we can visualize the biscuits and the friends accounts for the immediacy of the case in which the number of kilos of biscuits is greater than the number of friends. The linguistic control of the problem allows the clever student to notice the incongruence between the semiotic registers mentioned above when the number of students is greater than the number of biscuits units. The *referential, apophantic* and *discursive expansion* functions are crucial in organizing the distinctive features of the problem expressed in natural language, in order to perform the correct conversion into the arithmetical language and solve the problem through appropriate conversions. For a detailed analysis of the relation between intuitive thinking and semiotics we refer the reader to Andrà and Santi (2011, 2013).

Stereotype, intuitive model, parasitic model, textual semantics, skim-reading the problem statements, … how much more could be said.

Episode 3. Metacognition in the negative: "The fact is that I don't know …"
During the long night from the 25th to the 26th March 1993, in Sulmona (Abruzzo, Italy), Vergnaud and D'Amore discussed metacognition at length. D'Amore orally reports:

> For me it was a real lesson in didactics that I will never forget. And neither Gérard has forgotten it, since each time we meet he reminds me of it. In short, he claimed that the didactic purpose of metacognition should always be expressed in the negative: what I don't know, what I don't know how to do.

In fact, at that time the Bologna research team was investigating the "mathematics of time". One of the problems asked: "Antonio, the baker, works from 9:00 PM on Tuesday to 6:00 AM on Wednesday. How many hours does he work?".

Given at different school levels, it produced a great variety of results. But a little girl in second grade (primary school), after drawing a baker kneading the dough, wrote exactly the following words: "I don't know how to solve this problem and moreover I don't know how many hours are in the night". Of course, this "moreover" was understood as a "because".

A beautiful example of individual meaning that does not intersect the cultural one. She draws the baker kneading the dough, but she does not have a precise experience of how time passes during the night. The "9:00 PM to 6:00 AM" expression is meaningless to her, outside her zone of proximal development and her embodied experience. Natural language allows her to translate and objectify the distance between time in her personal meaning and time as a cultural object. A narrative use of natural language could build a bridge for the use of sexagesimal system, she already uses to indicate time during the day, also in the transition from 9:00 PM to 6:00 AM.

This is an excellent and significant example of conceptual metacognition in the negative, but extended to a procedural context: if I knew how many hours there are in the night, then I would know how to solve the problem. Awareness, mastery, metacognition. It seems to us that this episode contains a world of possible paradigms.

Episode 4. "Ah, but it can't be done that way"

The following problem suggested by E. Fischbein (1992) is well-known: "0.75 liters of orangeade cost 2 dollars; how much does 1 liter cost?". To this, one should respond with the banal operation $2 \div 0.75$. Instead, it gives rise to a great variety of surprising reactions, even more surprising when compared with the reactions obtained with the following, apparently entirely analogous exercise: "2 liters of orangeade cost 6 dollars; how much does 1 liter cost?". The same adults and students who respond to the second question with an immediate $6 \div 2$, almost never give the above division as their answer to the first question, looking instead for more semantically controlled ways such as the proportion: $0.75 : 2 = 1 : x$. Or they fall back on fractions, noting that 0.75 is 3/4. Having brought out how the second problem could be solved by calculating $6 \div 2$, we wondered if, by analogy, the students in 7th grade would accept that for the first problem one should calculate $2 \div 0.75$. But the response of more than one student can be condensed into the protest of one young man: "Ah, but it can't be done that way". So, the analogy does not come into play: the numerical datum 0.75 destroys a possible logical analogy between the two problems. We even tried asking:

"*A* liters of orangeade costs *B* dollars; how much does 1 liter cost?", suggesting that numbers could be used in place of *A* and *B* and proposing the generic solution $B \div A$. Of course, no one proposed anything but natural numbers. When I attempted to put 0.75 (or 0.5) in place of *A*, at the moment of redoing $B \div A$, the refusal mechanism sprang up once again: the "good" operations can no longer be used, and you have to fall back on other methods.

This is a paradigmatic example of non-congruence between the specific language of mathematics regarding division operations and the arithmetical language (idem). A conversion from the specific language used in the problem statement to the symbolic arithmetical language would allow us to solve the problem with simple calculations. The proportion establishes an immediate congruence between the problem statement and arithmetical symbolism. The symbol of division replaces the term "costs" and simple conversions and treatments make the problem apparently very easy. This turns out to be a trick that hides the lack of an adequate mediated reflexive activity, in the operational phase, to objectify division. The lack of a robust embodied objectification of division leaves us with a shallow and weak reification of this arithmetical operation and any semiotic transformation is meaningless. On the basis of an embodied meaning it is reasonable to foster a leap to higher levels of generality for division. The use of conversion and treatment with the aid of the discursive functions of natural language is at this point significant. Otherwise, students and teachers are left with stereotypes, prisoners of the strength of intuitive models: under these conditions resorting to proportions can be interpreted as the *formal delegation clause* of the didactical contract (D’Amore, 1999; D’Amore, Fandiño Pinilla, Marazzani, & Sarrazy, 2010; Narváez Ortiz, 2017). The structure of the problem is identical to the one in example 2, by substituting the kilos with the dollars and the friends with the liters. The same remarks apply to the role of natural language.

Stereotypes, mechanisms of analogy blocked by particular choices of numerical data, the role of data in the solution of a problem. Furthermore, intuitive models of division: indeed, if dividing means sharing out or containing, what could “*divide by 0.75*” possibly mean? And thus, once again, language.

We continue with other interesting verbal contributions (D’Amore, 1993c).

Episode 5. “The smallest number in the world”

Let us suppose we ask a second-grade student: “What is the smallest number in the world?”. Setting aside the reasonableness of the question and the formality of the linguistic register used, about which we will speak later, the answer “Zero” would be considered an optimal result. But what would we think if the same answer was given by students in eighth grade (13-14 years old), after they had come to know the set Z of the integers? And if the same answer was given by students in the final secondary school years, students who already knew mathematical analysis? And by fourth year undergraduates in a mathematics degree course? It would not be a big surprise if the question was asked in the particular linguistic and mathematical environment that is defined in D’Amore and Martini (1998) as the “natural context”, i.e. in a context of natural and not formal language.

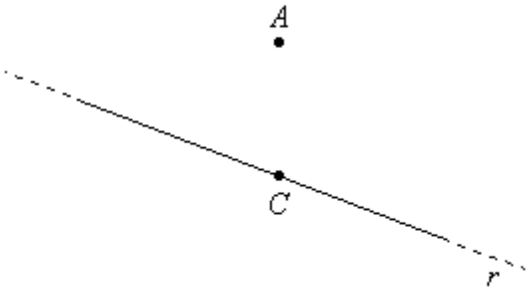
A series of questions of various kinds is posed in natural, colloquial language, by using the linguistic register of everyday language, totally ignoring the formal language that is normally used when mathematics is done in the classroom (and that is the reason behind the linguistic formality of the question given above). Among all these questions we require the use of natural language. In this way we create, subtly but apparently very strongly, an environment that we call the “natural context”, formed by everyday language and natural numbers. At this point, the response “Zero” to the preceding question becomes clear. (It might be interesting to know that many students in the final secondary school years who have studied mathematical analysis and very many fourth-year mathematics undergraduates answered writing “$-\infty$”. Perhaps it should also be said that the fourth year of the mathematics degree course where we conducted this experiment was not housed in an Italian university). But the story does not end here. The next question was: "Which is smaller, –3 or +2?". This question breaks out of the natural context … And nevertheless, there are many of the students who answer "+2" here, while leaving "Zero" for the preceding question. Others, however, go back and modify. (We must also say, in the interests of truth, that many students know that, in view of the existence of the negative numbers, the answer "Zero" is incorrect and is not good enough, but they do not know what to write. Those who reach the point of writing, in words, "doesn't exist", "there's no such thing", or something similar, are in a tiny minority, even among the primary school teachers interviewed).

The notion of "big" and "small" is exactly at the point where personal/embodied meaning and cultural/disembodied meaning crash against each other. *Small* in what sense? In terms of absolute value, in terms of number ordering, or in terms of limit? "$-\infty$" is big or small? The embodied notion of big and small needs to be transformed at a higher level of generality in the context of numbers. Furthermore, the structure of the semiotic register undergoes important changes that hinder an invariant notion of big and small as we move from natural numbers to integers. The presence of the plus and minus signs puzzles the students and mixes up the possible meanings of "big" and "small" mentioned above. The *referential*, *apophantic, discursive expansion* and *discursive reflexivity* functions in the context of number systems allow us to designate numbers in the different number systems, formulate claims regarding "big", "small" etc., organize a discourse and recognize the validity, the mode and the status of the claims. The coordination of natural language, specific mathematical language, and symbolic language outlines the correct meaning of *big/biggest* and *small/smallest* "number in the world".

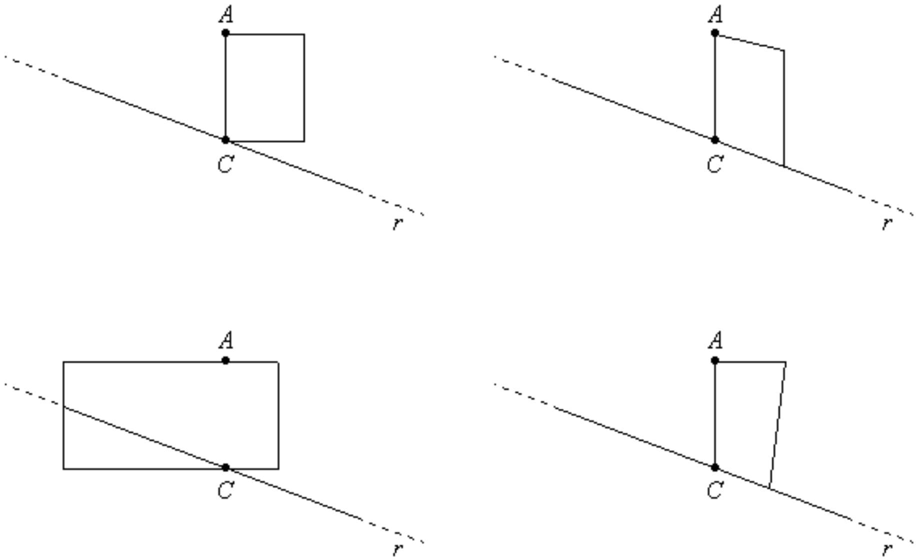
Here one might reflect upon the reasonableness of contexts, upon the influence that these have on mathematics classroom, upon artificial environments, upon the blend of natural and mathematical language, and once again upon stereotypes and intuitive numerical models.

Episode 6. "My God!"

Here is the account of an experiment carried out in a foreign country (outside of Italy) with upper secondary school students (pupils aged 15 to 20) (Gallo, Amoretti & Testa, 1989, p. 14): "Draw the rectangle *ABCD* with side *BC* along the straight-line *r*".



And here are some of the responses (reproduced at a greatly reduced scale: the actual originals are available for those who would like to see them).



One of the mathematics teachers simply said: "My God!", when we showed him the results obtained in *his* class. The discussion of the causes that underlie these results is well carried out in (Gallo, Amoretti, Testa, 1989, p. 14). But, in addition, we wanted to study the relationship between these responses and the conceptual difficulty of understanding the sense of the request in all its complexity, the problem of the use of stereotypes with regard to geometric figures, according to which rectangles have horizontal bases etc. In particular, on this occasion, our discussion with these teachers covered natural language and the non-neutrality of the language in which geometry is done and geometrical problems are presented.

Objectification of geometrical figures is usually embodied in the perceptual space of the student. The individual's experience of space is strongly influenced by the fact that we are immersed in gravity. He feels a sense of proximity towards geometrical entities when semiotics means of objectification recall the vertical structure of our experience. The specific language of geometry and natural language tend to reinforce the relationship between the positioning of figures in the geometrical space and the embodied experience of physical space: "base", "height", "the horizontal and vertical lines", for example. When the individual meaning of geometrical figures has to be aligned with their cultural meaning, the student lives a rupture in neglecting a privileged position according to the verticality of physical space. The use of linguistic terms such as "horizontal" and "vertical", "base" and "height" do not allow to objectify the geometrical object at higher levels of generality. The use of natural language hinders an appropriate learning of geometry and can result in stereotypes and parasitic intuitive models. The solution of the problem requires a leap to higher levels of generality that allow the students to go beyond their perceptual experience of space. The solutions presented above show the strength and the need for embodied meaning on the part of the students. In fact, the students quit any logical control of the problem and give nonsense answers, a behavior typical of the didactical contract. The answers are nonsense as regards the institutional mathematical meaning, but they make sense at their personal level. The students are aware that their figures are not rectangles, or they do not accomplish the request of the problem, but they make sense in their effort, trapped in a parasitic strong model, to attend the request of the problem. The challenge of teaching and learning mathematics is to push students towards higher levels of generality without losing the core of their individual embodied experience. Once again, natural language can be a resource to objectify figures in the geometrical space starting from the physical one.

This is a significant example that questions our understanding of perception. Often one believes that perception is just an intentional movement of consciousness towards a physical or mental object that exists per se. Instead the object's mode of existence is entangled with the way our intentional act attends it. The problem for the student is to attend to the figure according to the constraints of the problem. There is no a priori rectangle that the students are unable to see. There are different modes of existence of the rectangle corresponding to the different intentional ways to attend them. The point is that the students' intentional way of referring to the geometrical figure is different from the cultural and general one required for the solution of the problem. Furthermore, the student's way of attendance is tainted by schemes, beliefs and interpretations. As some of the above drawings show, some students are convinced they have to connect points A and C to form a side of the rectangle. The educational issue is how to "domesticate" the student's eye to attend geometrically the drawing proposed in the problem statement, i.e., borrowing Radford's terminology, how to transform the biological eye in "the eye as a theoretician". For an in-depth analysis of this topic we refer the reader to Radford (2010). The eye is culturally “domesticated” also with the help of natural language that can support the transition from the experience of the physical space to the objectification of the geometrical one. When moving towards higher levels of generality, the student broadens the meaning of terms such as “height”, “vertical”, “bottom” and “top” with terms that objectify the rectangle independently from our physical perception of objects. For example, the term “height” becomes the “side” of the rectangle and the term “vertical” becomes the “angle between one of its sides and a straight line”. When the students are able, supported by the natural language, to imagine the rectangle as a specific relation between four segments and all its possible positions in the plane, the above problem becomes immediate.

In terms of representation registers there is no direct correspondence – moreover such a correspondence is biased by a strong intuitive model of the rectangle – between the problem statement in natural language and the geometric figural representation register. The correct conversion would make the problem extremely simple. The discursive expansion and the discursive reflexivity are crucial in erasing the bias mentioned above, recognizing the distinctive features of the rectangle in the context of this problem, and establishing the correct conversion. This is a good example that shows that the semiotic control is one of the possible and necessary learning in mathematics (Bolondi & Fandiño Pinilla, 2008). Is it a mathematical, a semiotic or a linguistic issue? Where do we draw the boundaries between the three domains?

Episode 7. “You can’t do that!”

Very rapidly: primary school, we propose an impossible problem. The children all answer by simply adding the numerical data in the statement of the problem (a similar thing is reported in D’Amore, 1993b). We explain to the children that the problem is impossible. Nervous sniggers; the most impetuous youngster objects: “Ah, but you can’t do that! If the problem is impossible you should have said so. Our teacher does”. Yes, this is certainly the didactic contract and the transparency clause. Yes, this is certainly the *Topaze effect* (Brousseau, 1997; D’Amore, Fandiño Pinilla, Marazzani, & Sarrazy, 2010; Narváez Ortiz, 2017). But it also concerns the general model of a problem and stereotypes (Zan, 1991-1992).

Impossible problems, also known as the *“age of the captain effect”* (Chevallard, 1980; D’Amore, 1999), are effective in unveiling the student’s beliefs, interpretations and expectations that contribute to their personal meaning regarding mathematics. Here, a problem in mathematics *must have* a solution, possibly obtained after some calculation. The teacher’s mathematical practice has been reified by the students into a strong operational invariant: “Given a problem, I have to find the solution”. When this expectation is deceived the students react: “You can’t do that!”: Brousseau’s didactical contract at its climax. Natural language can support new narratives that trigger forms of metacognition in which the individual can recognize the beliefs and interpretations that trap their mind and find emotional and cognitive strategies to overcome them.

Episode 8. "When the signs are different, you can't simplify"

We and a couple of friends who teach at the professional institutes in Bologna (Italy) are analyzing the "hidden curriculum". The basic idea is that the inappropriateness and the incoherence of the students' responses are sometimes global rather than local. In other words, the student sometimes makes a mistake because he is following some rule of his own that he has used for years. At times it has worked, and so he has expressed, generalized and accepted it. After that, he stubbornly follows it and does not understand why the teacher sometimes accepts it and sometimes not. In the context of algebra, we apply the "Pretend you are a teacher" trick. The student has to indicate which simplifications are correct and which are not from a list. One student, for example, recognizes that the simplification:

$$\frac{\cancel{x^2}+5}{\cancel{x^2}-5}$$

is incorrect, in short: you can't do that. Bravo, that's right. But, when asked why, he responds: "Because there are two different signs". This raises a doubt in our minds, and so we propose:

$$\frac{-\cancel{5}}{+\cancel{5}}$$

You can't do that either, *for the same reason*. Thus, the student has created for himself a rule described in words in a rather ambiguous fashion (yes, we know: that's a euphemism); on some occasions it works well, and the teacher praises him, on other occasions it does not, and he does not understand the reason why. The student is locally coherent, as he follows this rule of his in a context in which it is globally incoherent. We understand this, he does not. All he understands is that something is not working, but he does not know how to explain the reason to himself. The set of rules, both those that are correct and those that are not (from an adult point of view), constitutes a "hidden curriculum" that is the true paradigm of the student's algebraic behavior.

Another example that shows how learning results in a shallow understanding of mathematics without an underlying mediated reflection. The student, under Duval's cognitive paradox, is compelled to identify the mathematical object with the algebraic semiotic representation. The unavoidable lack of meaning is filled with operational invariants derived from the belief that mathematical meaning lies in some kind of symbolic manipulation. This behavior is driven, within the didactical contract, by the *clause of formal delegation* and the *clause of formal justification demand*

(D'Amore, 1999; D'Amore, Fandiño Pinilla, Marazzani, & Sarrazy, 2010; Narváez Ortiz, 2017). A linguistic control of the situation through the *discursive expansion* and *discursive reflexivity* functions could be an effective instrument to overcome the cognitive paradox and ascribe mathematically correct meaning to algebraic fractions. Too often, in high school, algebra is introduced immediately through symbols without the support of the discursive functions of natural language.

4. Conclusions

Mathematical thinking and learning is one of the most challenging and fascinating intellectual adventures. Nevertheless, it requires cognitive and metacognitive skills that allow students to master the broad and composite semiotic activity beneath the development of mathematics. We have shown the role that the natural language can have in mastering the semiotics that characterizes mathematical thinking and learning.

Natural language in TKO, synchronically used with other semiotic means of objectification, plays a key role in driving reflexive mediated activity. It entails a set of possibilities that are the indelible basis for the recognition of fixed patterns of reflexive activity, thus accessing higher layers of generality. The discursive functions of natural language (in a structural and functional framework) are crucial, as a metacognitive support, in recognizing the distinctive traits of mathematical objects, in expanding reasoning and assigning validity and status to claims. Natural language is a pivotal representation register for treatment and conversion operations.

Natural language can hinder the learning of mathematics when it discards the pupils personal meaning, made of embodied experience, feelings, emotions, interpretations, in its effort to encounter the cultural/institutional one. It is most interesting to see how students use the everyday language and feel no need to fall back on artificial or fabricated languages, even when they want to talk about mathematics. Could it be that this is an entirely adult need? If so, then this should become more expected and it should prove the Italian governmental primary school mathematics national curriculum (dating back to 1985, today no longer in force) to be correct, where they say: "(…) natural language has expressive richness and logical potential that are suited to the needs of learning".

Natural language is the privileged context for communication as far as each individual is concerned. Denying this hinders the interpretation of students' responses. Admitting it always provides a key to an extremely interesting reading. For many years we have been experimenting with the opportunity of doing as much mathematics as possible in Italian and not in *mathematics jargon* (D'Amore, 1993c). Our students already have a hard job using their everyday language well; demanding additional sophisticated subtleties can be harmful (even more than pointless). Many other researchers agree with this position, for example Hermann Maier, who has devoted considerable efforts to this issue (see, for example: Maier, 1996). The fact is that some teachers think that natural language is not suitable for doing mathematics. Of course, our objectives must also include that of *arriving* at a perfect understanding of the use of mathematical language; but as an *educational objective*, not as a prerequisite. And besides, especially with younger students, more than doing mathematics, we should be talking about mathematics.

All of us – and this is a law of the pragmatics of human communication – along with the messages we send out consciously, also provide messages that we do not want to provide but which the intended hearer of the message receives, with more or less explicit awareness. This takes place even in the most beautiful and perfect mathematics lessons in the world. Knowledge of the intuitive models we are inducing, perhaps unconsciously, in the students is of the greatest importance when we seek to understand what are the conceptual schemes that the students themselves make of the models in place of those we wanted to provide. The matter is rather complex, and we refer you to D'Amore and Frabboni (1996) for a much more exhaustive treatment. The teacher – who simply believes that the model put forward in the lesson coincides with the one formed in the head of the students to whom it is communicated – is wrong, highlighting quite a lot of naivety. Knowing that matters of this kind are problematic removes astonishment towards certain answers of our students.

Stereotypes are just unavoidable. Combating them is one of the main jobs of the teacher. They are sneaky mental creatures always lying in wait. It is incredible how it only takes two or three examples that agree in some insignificant way before the student generalizes them in operational invariant, creating stereotypes. Culturally speaking, the student is a conservative who tends to jump at forming rules and models for everything: "So, every time you get zero it means that the equation is indeterminate": claimed a student one day when we had worked and discussed a single example of an equation with him. A single example, and he had already created a rule for all cases. If we had not immediately worked out a counterexample for him, he would have registered the rule, with easy-to-imagine harmful effects on the next assignment. We observe, *en passant*, that this boy is an intelligent, critical young man, always ready to discuss, object and quibble. Except in mathematics, where he immediately forms stereotypes in order to reassure himself. Ah, what an image of mathematics he must have made over the years. No, we are not straying from the subject: the image of mathematics, the image of oneself doing mathematics, rules to follow without motivation (apart from that of getting a good grade), stereotypes, they all go hand in hand. They are different – but not all that different – facets of the same problem.

Teaching mathematics needs to know mathematics well and in depth. But to have success in the process of learning mathematics on the part of one's own students, at whatever level of schooling, perhaps knowing mathematics is no longer enough. One needs to know the operating mechanisms of that complex machine that allows learning to take place. Since in the current state of knowledge this is impossible we ought at least to know those aspects that reach the surface, study them, understand them. Conducting research on these aspects is the best way to come to grips with them.

Acknowledgements

The authors would like to thank the reviewers for improving the original version of the article and PhD Maura Iori for thoroughly editing the manuscript and for her critical contributions that enhanced the theoretical framework.

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